On the prime number theorem , a new original proofs on the analytic theory of numbers , Unveiling $p_n \approx nln(n)$

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Abstract

Presenting a novel approach, this paper offers a captivating proof of the prime number theorem's approximation $p_n \approx nln(n)$. By creatively intertwining complex analysis, Fourier techniques, and Dirichlet series, our work reveals the enchanting connection between prime distribution and the Riemann zeta function, emphasizing the aesthetic beauty of this analytical relationship.

Zeta function relation with prime numbers :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}} \tag{1}$$

-Proof :

We have that:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} (= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + ...)$$

let:

$$p \in \mathbb{P} :$$

$$\frac{1}{p^s}.\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(np)^s} = \sum_{k \in M_p} \frac{1}{k^s}$$

such that:

$$M_p := \{n \in \mathbb{N} : p|n\} = p\mathbb{Z} \cap \mathbb{N}$$

therefore

$$\zeta(s) - \frac{1}{p^s}\zeta(s) = \sum_{n \in \mathbb{N}^* : n \notin M_p} \frac{1}{n^s} = \zeta(s)(1 - \frac{1}{p^s})$$

So using this operation we subtract all terms of multiples of p for example let's take that p = 3 then :

$$(1 - \frac{1}{3^s})\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{5^s} + ..$$

so let's subtract also the multiples of 2 :

$$\rightarrow (1 - \frac{1}{2^s})(1 - \frac{1}{3^s})\zeta(s) = \sum_{n \in \mathbb{N}^* : n \notin M_2 \cup M_3} \frac{1}{n^s}$$

Repeating for the next terms ... : until we get :

$$(\prod_{p \in \mathbb{P}} 1 - \frac{1}{p^s})\zeta(s) = \sum_{n \in \mathbb{N}^* : n \notin \bigcup_{i \in \mathbb{N}^*} M_i} \frac{1}{n^s} \qquad (2)$$

therefore :

$$(\prod_{p\in\mathbb{P}} 1-\frac{1}{p^s})\zeta(s)=\sum_{n\in\mathbb{N}^*:n\in\bigcap_{i\in\mathbb{N}^*}\bar{M}_i}\frac{1}{n^s} \qquad (3)$$

and we know that :

$$\bigcap_{i\in\mathbb{N}^*}\bar{M}_i=\bigcap_{i\in\mathbb{N}^*}\{n\in\mathbb{N}^*|n\not\equiv 0[i]\}=1 \qquad (4)$$

because :

$$(\forall n\in\mathbb{N}^*-\{1\})(\exists p\in\mathbb{P}):p|n$$

$$\therefore (\prod_{p\in\mathbb{P}} 1-\frac{1}{p^s})\zeta(s)=1\Rightarrow\zeta(s)=\prod_{p\in\mathbb{P}}\frac{1}{1-\frac{1}{p^s}} \qquad (5)$$

1 proof using induction

I was thinking If I could test euler product formula using induction while that $\mathbb{N}$ is countable set. , suddenly I coincide with an elegant proof which I would to provide it in this section.
let p_n be the nth prime.
here is an equivalent formulation of (1) (sieve method)

$$\zeta(s)=\prod_{n=1}^{\infty}(\frac{1}{1-\frac{1}{p_n}}) \qquad (6)$$

Lemma 1.

$$\prod_{i=1}^{n}(1-\frac{1}{p_i^s})\zeta(s) = \sum_{gcd(i,\prod_{i=1}^{n}p_i)=1}\frac{1}{i^s}$$

.

Proof. for $n = 1$: we have that (trivial) :

$$\zeta(s)-\frac{1}{2^s}\zeta(s) = \sum_{n\in\mathbb{N}^*:n\notin M_2}\frac{1}{n^s} = \zeta(s)(1-\frac{1}{2^s}) = \sum_{gcd(i,2)}\frac{1}{i^s}$$

let $n \in \mathbb{N}$: suppose that :

$$\prod_{i=1}^{n}(1-\frac{1}{p_i^s})\zeta(s) = \sum_{gcd(i,\prod_{i=1}^{n}p_i)=1}\frac{1}{i^s}$$

$\therefore$

$$(1-\frac{1}{p_{n+1}})\prod_{i=1}^{n}(1-\frac{1}{p_i^s})\zeta(s) = \sum_{gcd(i,\prod_{i=1}^{n}p_{n+1}p_i)=1}\frac{1}{i^s}$$

$\Longrightarrow$

$$\prod_{i=1}^{n+1}(1-\frac{1}{p_i^s})\zeta(s) = \sum_{gcd(i,\prod_{i=1}^{n+1}p_i)=1}\frac{1}{i^s} \qquad (7)$$

taking the limite of (7) :

$$\prod_{i=1}^{\infty}(1-\frac{1}{p_i^s})\zeta(s) = 1$$

$\therefore$

$$\zeta(s) = \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p^s}}$$

$\square$

Proof of infinitely many primes using Zeta Euler product

as we proved before that

$$\zeta(s) = \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p^s}} \Rightarrow \zeta(1) = H_\infty = \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p}} = \infty$$

such that

$$(\forall n \in \mathbb{N}^*) : H_n := \sum_{k=1}^{n} \frac{1}{n}$$

(the harmonic sequence)

$$\therefore \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p}} = \infty$$

Suppose there exist a finite number of primes : We know that

$$\forall p \in \mathbb{P} : p > 1$$

so the product:

$$\prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p}}$$

is a finite product because:

$$\forall p \in \mathbb{P} : p > 1 \Rightarrow 0 < 1 - \frac{1}{p} < 1$$

and while that the product :

$$\prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p}}$$

is finite and

$$1 - \frac{1}{p} \nsim 0 \Rightarrow \prod_{p\in\mathbb{P}} \frac{1}{1-\frac{1}{p}} \nsim \infty$$

Absurd because the Harmonic series is a divergent . I know , that you don't like the proof above therefore I intended to prove it to you using Euler formulation :

$$\sum_{p\in\mathbb{P}} \frac{1}{p} = \infty \tag{8}$$

Proof. we have that :

$$\prod_{p\in\mathbb{P}} \frac{p}{p-1} = \infty$$

(using euler product in $\zeta(1)$)
$\therefore$

$$\sum_{p\in\mathbb{P}} ln(\frac{p}{p-1}) = \infty$$

$\Longrightarrow$

$$-\sum_{p\in\mathbb{P}} ln(\frac{p-1}{p}) = \infty$$

implies

$$-\sum_{p\in\mathbb{P}} ln(1-\frac{1}{p}) = \infty$$

$\therefore$

$$\sum_{p\in\mathbb{P}} \frac{1}{p} = \infty$$

$\because$

$$\frac{1}{p_n} \approx ln(1+\frac{1}{p_n})$$

observe that $\lim p_n = \infty$ $\square$

2 is there any equivalent to the sequence of prime p_n ?

Theorem 2. *(Prime Number Theorem (GAUSS PNT))*

$$\pi(n) \approx \frac{n}{ln(n)} \tag{9}$$

Remark :

$$\pi(n)_{n\in\mathbb{N}^*} \overset{def}{=} Card(\{p \leq n | p \in \mathbb{P}\})$$

Lemma 3. *let* $u_n, v_n \in \mathbb{R}^{\mathbb{N}}$ *such that :* $u_n \approx v_n$ *and* $A_n \in \mathbb{R}^{\mathbb{N}}$ *such that :* $\lim A_n = \infty$
$\therefore u_{A_n} \approx v_{A_n}$

Proof. let $\epsilon > 0$ let's prove that :
$\exists N \in N \ \forall n \in \mathbb{N} \ n \geq N \implies |\frac{u_{A_n}}{v_{A_n}} - 1| < \epsilon$
we have by definition :
$\exists p \in \mathbb{N} \ \forall n \in \mathbb{N} \ n \geq p \implies |\frac{u_n}{v_n} - 1| < \epsilon$
and by definition :
$\exists p_0 \in \mathbb{N} \ \forall n \in \mathbb{N} \ n \geq p_0 \implies A_n > p$
therefore by taking $N = p$ we find the concerning definition.
$\exists N \in N \ \forall n \in \mathbb{N} \ n \geq N \implies |\frac{u_{A_n}}{v_{A_n}} - 1| < \epsilon$
QED. □

Corollary 3.1.

$$n \approx \frac{p_n}{ln(p_n)}$$

Proof. using the $[Lemma 3.0]$ and $[thm 2]$ we find
$\pi(p_n) \approx \frac{p_n}{ln(p_n)}$
and by definition we have that : $\pi(p_n) = n$
which give us the corollary. QED. □

Corollary 3.2.

$$ln(p_n) \approx ln(n)$$

Proof. $[Cor 3.1]$ give us that

$$\frac{p_n}{ln(p_n)} = n(1 + o(1))$$

then

$$ln(p_n) - ln(ln(p_n)) = ln(n) + ln(1 + o(1))$$

therefore

$$ln(p_n) + o(ln(p_n)) = ln(n) + o(1)$$

which give us

$$ln(p_n) \approx ln(n)$$

QED. □

Corollary 3.3.

$$p_n = n^{1+o(1)}$$

Proof. $[Cor3.2]$ give us

$$ln(p_n) = ln(n)(1 + o(1))$$

therefore

$$p_n = \exp(ln(n))^{1+o(1)}$$

which give us the final result

$$p_n = n^{1+o(1)}$$

in other part : $\exists \epsilon_n \in \mathbb{R}^{\mathbb{N}}$ such that $\lim \epsilon_n = 0$ and $p_n = n^{1+\epsilon_n}$

□

without giving estimations , let's find a simple equivalence of n^{ϵ_n}

Corollary 3.4.

$$p_n \approx nln(n) \tag{10}$$

Proof. $[Cor3.1]$ and $[Cor3.3]$

$$n^{1+\epsilon_n} \approx nln(p_n)$$

therefore

$$n^{\epsilon_n} \approx ln(p_n)$$

as we have been seen in $[Cor3.2]$

$$p_n \approx ln(n)$$

by transitivity of $\approx$

$$n^{\epsilon_n} \approx ln(n)$$

and as we find before in proof of $Cor[3.3]$

$$p_n = n^{1+\epsilon_n} = n.n^{\epsilon_n} \approx n.ln(n)$$

$\therefore$

$$p_n \approx nln(n)$$

which give us the corollary. QED. □

we could observe simply that ,

$$\sum_{p \in \mathbb{P}} \frac{1}{p}$$

is divergent.
because we know that the bertrand serie :

$$\sum_{n=2}^{\infty} \frac{1}{nln(n)}$$

is divergent.